

MODULAR INVARIANCE AND TWISTED ANOMALY CANCELLATIONS FOR CHARACTERISTIC NUMBERS

QINGTAO CHEN AND FEI HAN

ABSTRACT. By studying modular invariance properties of some characteristic forms, we obtain twisted anomaly cancellation formulas. We apply these twisted cancellation formulas to study divisibilities on spin manifolds and congruences on spin^c manifolds. Especially, we get twisted Rokhlin congruences for $8k + 4$ dimensional spin^c manifolds.

1. INTRODUCTION

Let M be a 12 dimensional smooth Riemannian manifold. A beautiful relation between the top degree components of the Hirzebruch \hat{L} -form and \hat{A} -form of M was shown by Alvarez-Gaumé and Witten [1] as a gravitational anomaly cancellation formula as follows,

$$(1.1) \quad \left\{ \hat{L}(TM, \nabla^{TM}) \right\}^{(12)} = \left\{ 8\hat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - 32\hat{A}(TM, \nabla^{TM}) \right\}^{(12)},$$

where $T_{\mathbf{C}}M$ denotes the complexification of TM and $\nabla^{T_{\mathbf{C}}M}$ is canonically induced from ∇^{TM} , the Levi-Civita connection associated to the Riemannian structure of M ; $\text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M})$ denotes the Chern character form associated to $(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M})$ (cf. [22]). This gravitational anomaly cancellation formula, which they called miraculous cancellation formula, was derived from very non-trivial computations.

(1.1) is generalized by Kefeng Liu [16] to arbitrary $8k + 4$ dimensional manifolds by developing modular invariance properties of characteristic forms. In [16], he proved that for each $(8k + 4)$ -dimensional smooth Riemannian manifold M the following identity holds,

$$(1.2) \quad \left\{ \hat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M).$$

In (1.2), each $h_r(T_{\mathbf{C}}M)$ is a differential form

$$\left\{ \hat{A}(TM, \nabla^{TM}) \text{ch} \left(b_r(T_{\mathbf{C}}M), \nabla^{b_r(T_{\mathbf{C}}M)} \right) \right\}^{(8k+4)},$$

where $b_r(T_{\mathbf{C}}M) \in KO(M) \otimes \mathbf{C}$, $0 \leq r \leq k$, can be derived canonically from TM . When the manifold is closed and spin, according to the Atiyah-Hirzebruch divisibility [3], $\langle h_r(T_{\mathbf{C}}M), [M] \rangle$ are all even numbers. Therefore from (1.2) and the Hirzebruch signature theorem [11], we easily get the Ochanine divisibility [19], which says that the signature of an $8k + 4$ dimensional smooth closed spin manifold

Date: May 10, 2006.

1991 Mathematics Subject Classification. Primary 53C20, 57R20; Secondary 53C80, 11Z05.

is divisible by 16. This shows us how miraculous cancellation formulas imply divisibility of characteristic numbers. The author also provides a similar miraculous cancellation formula for $8k$ dimensional manifolds. Unfortunately this $8k$ dimensional cancellation formula does not imply divisibility results. Note that in some sense, Liu's formula refines the arguments of Hirzebruch [10] and Landweber [14], who deduce the Ochanine divisibility by using the ideas of elliptic genus, to the level of differential forms. See [16] for details.

Liu's method is taken over in [8, 9] to study the Ochanine congruence and the Finashin congruence. The authors show that there actually exist more general miraculous cancellation formulas with an extra complex line bundle involved, which turn out to be efficient to study congruence phenomena on spin^c and pin^- manifolds. Let (ξ, ∇^ξ) be a real oriented Euclidean plane bundle, or equivalently a complex line bundle on M with Euler form $c = e(\xi, \nabla^\xi)$. For each $8k+4$ dimensional smooth Riemannian manifold M , they obtain that

$$(1.3) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8k+4)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

In (1.3), each $h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$ is a differential form

$$\left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} \left(b_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}), \nabla^{b_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})} \right) \cosh \left(\frac{c}{2} \right) \right\}^{(8k+4)},$$

where $\xi_{\mathbf{C}}$ is the complexification of ξ and $b_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \in KO(M) \otimes \mathbf{C}$, $0 \leq r \leq k$, can be derived canonically from TM and ξ . (1.3) is a generalization of Liu's miraculous cancellation formula (1.2) in the sense that when $c = 0$ it exactly gives Liu's formula. Especially on dimension 12, it gives a generalization of the Alvarez-Gaumé-Witten miraculous cancellation formula,

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(12)} = \left\{ \left[8\widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - 32\widehat{A}(TM, \nabla^{TM}) \right. \right. \\ \left. \left. - 24 \widehat{A}(TM, \nabla^{TM}) (e^c + e^{-c} - 2) \right] \cosh \left(\frac{c}{2} \right) \right\}^{(12)}.$$

As an application of the general cancellation formula (1.3), when M is closed and spin^c and B is an $8k+2$ dimensional oriented submanifold of M such that $[B] \in H_{8k+2}(M, \mathbf{Z})$ is dual to $w_2(TM)$, the authors obtain that $(\text{sign}(M) - \text{sign}(B \bullet B))$ is divisible by 8 by using the Atiyah-Singer index theorem for spin^c manifolds, where $(B \bullet B)$ denotes the self-intersection of B in M . Moreover, they [9] show that

$$(1.4) \quad \frac{\text{Sig}(M) - \text{Sig}(B \bullet B)}{8} \\ \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_k(T_{\mathbf{C}}M + \mathbf{C}^2 - \xi_{\mathbf{C}}, \mathbf{C}^2)) \cosh \left(\frac{c}{2} \right) \\ \equiv \text{ind}_2(b_k(TB + \mathbf{R}^2, \mathbf{R}^2)) \pmod{2},$$

which is the analytic version of the Ochanine congruence obtained in [18]. Formula (1.3) has interesting applications to study the Ochanine and the Finashin congruences (cf. [19], [6]). We refer interested readers to [9] for details. This shows us again how miraculous cancellation formulas imply divisibility and congruence results.

Looking at these miraculous cancellation formulas as well as the divisibilities and congruences induced by them, one naturally asks if there exist more miraculous cancellation formulas like (1.2) and (1.3) and consequently exist more divisibilities and congruences for characteristic numbers. We show in this article that the answer is positive.

To be more precise, still applying modular invariance of characteristic forms [16], we obtain some interesting twisted anomaly cancellation formulas. When these new cancellation formulas are applied to $8k$ and $8k + 4$ dimensional closed spin manifolds, we find some hidden divisibilities of the characteristic numbers $\langle \widehat{L}(TM) \text{ch}(T_{\mathbb{C}}M), [M] \rangle$, $\langle \widehat{L}(TM) \text{ch}(T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M), [M] \rangle$ and some of their linear combinations. The divisibilities of the characteristic number $\langle \widehat{L}(TM) \text{ch}(T_{\mathbb{C}}M), [M] \rangle$ for $8k$ and $8k + 4$ dimensional spin manifolds were already obtained by Hirzebruch [12] by studying elliptic genera. Our cancellation formulas supply an interesting approach to prove the Hirzebruch divisibilities. Moreover we are able to construct examples to show that the Hirzebruch divisibilities are best possible. On the other hand the divisibilities of the characteristic number $\langle \widehat{L}(TM) \text{ch}(T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M), [M] \rangle$ for $8k$ and $8k + 4$ dimensional spin manifolds induced by our cancellation formulas look new in the literature. We are also able to construct examples to show that these divisibilities are best possible. Note that these characteristic numbers are the indices of the elliptic operators $d_s \otimes T_{\mathbb{C}}M$ and $d_s \otimes T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M$ respectively, where d_s is the signature operator. The twisted signature operator $d_s \otimes T_{\mathbb{C}}M$ is already proved to be rigid by using Witten rigidity theorem, however nobody has been able to give a direct proof without using it [17]. Our divisibilities are still applications of modular invariance. It would also be interesting to find out a direct proof of them.

When we apply our cancellation formulas with a twisted complex line bundle to spin^c manifolds, we obtain some congruence results about the characteristic number $\langle \widehat{L}(TM) \text{ch}(T_{\mathbb{C}}M), [M] \rangle$, which in dimension $8k + 4$ give twisted Rokhlin congruence formulas.

The rest of the article is organized as follows. We list the twisted anomaly cancellation formulas in Section 2 and postpone their proofs to Section 5. In Section 3, we apply our twisted anomaly cancellation formulas to spin manifolds and obtain divisibilities for the tangent twisted signature. Then in Section 4, the twisted anomaly cancellation formulas are applied to spin^c manifolds and particularly induce twisted Rokhlin congruence formulas for $8k + 4$ dimensional spin^c manifolds.

2. TWISTED ANOMALY CANCELLATION FORMULAS

In this section, we first present some basic geometric data and then list the twisted miraculous cancellation formulas.

Let M be a smooth Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection and $R^{TM} = \nabla^{TM,2}$ the curvature of ∇^{TM} . ∇^{TM} extends canonically to a Hermitian connection $\nabla^{T_{\mathbb{C}}M}$ on $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$.

Let $\widehat{A}(TM, \nabla^{TM})$, $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined by

$$(2.1) \quad \widehat{A}(TM, \nabla^{TM}) = \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh \left(\frac{\sqrt{-1}}{4\pi} R^{TM} \right)} \right),$$

$$(2.2) \quad \widehat{L}(TM, \nabla^{TM}) = \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh \left(\frac{\sqrt{-1}}{4\pi} R^{TM} \right)} \right).$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E = \nabla^E \cdot \nabla^E$ (resp. $R^F = \nabla^F \cdot \nabla^F$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\text{ch}(G, \nabla^G) = \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^E \right) \right] - \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^F \right) \right].$$

In the rest of the paper, where there will be no confusion about the Hermitian connection ∇^E on Hermitian vector bundle E , we will write simply $\text{ch}(E)$ for the associated Chern character form.

Let ξ be a rank two real oriented Euclidean vector bundle, or equivalently a complex line bundle, over M carrying a Euclidean connection ∇^ξ .

If E is a complex vector bundle over M , set $\widetilde{E} = E - \mathbf{C}^{\text{rk}(E)}$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half complex plane.

We introduce four elements (cf. [16], [9]) in $K(M)[[q^{\frac{1}{2}}]]$ which consist of formal power series in $q^{\frac{1}{2}}$ with coefficients in the K -group of M ,

$$(2.3) \quad \Theta_1(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbf{C}}M}),$$

$$(2.4) \quad \Theta_2(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M}),$$

$$(2.5) \quad \Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbf{C}}M} - 2\xi_{\mathbf{C}}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}),$$

$$(2.6) \quad \Theta_2(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M} - 2\xi_{\mathbf{C}}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbf{C}}}).$$

Recall that for an indeterminate t ,

$$(2.7) \quad \Lambda_t(E) = \mathbf{C}|M + tE + t^2\Lambda^2(E) + \cdots, \quad S_t(E) = \mathbf{C}|M + tE + t^2S^2(E) + \cdots,$$

are respectively the total exterior and symmetric powers of E . The following relations hold between these two operations (cf. [2]),

$$(2.8) \quad S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}.$$

We can formally expand these four elements into Fourier series in $q^{\frac{1}{2}}$,

$$(2.9) \quad \Theta_1(T_{\mathbf{C}}M) = A_0(T_{\mathbf{C}}M) + A_1(T_{\mathbf{C}}M)q^{\frac{1}{2}} + \cdots,$$

$$(2.10) \quad \Theta_2(T_{\mathbf{C}}M) = B_0(T_{\mathbf{C}}M) + B_1(T_{\mathbf{C}}M)q^{\frac{1}{2}} + \cdots,$$

$$(2.11) \quad \Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = A_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) + A_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})q^{\frac{1}{2}} + \cdots,$$

$$(2.12) \quad \Theta_2(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = B_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) + B_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})q^{\frac{1}{2}} + \cdots,$$

where the A_i 's and B_i 's, are elements in the semi-group formally generated by Hermitian vector bundles over M . Moreover, they carry canonically induced Hermitian connections.

Let $c = e(\xi, \nabla^\xi)$ be the Euler form of ξ canonically associated to ∇^ξ .

Now we can state our twisted cancellation formulas and discuss their applications in the following subsections.

Theorem 2.1. *For $8k + 4$ dimensional smooth Riemannian manifold M , the following identity holds,*

$$(2.13) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\ = 2^{14} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M) \right],$$

where each $h_r(T_{\mathbf{C}}M) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbf{C}}M)) \right\}^{(8k+4)}$, $0 \leq r \leq k$, and each $b_r(T_{\mathbf{C}}M)$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M)$, $0 \leq j \leq r$. The right hand side is understood as 0 when $k < 1$.

In Theorem 2.1, putting $k = 0, 1, 2$, by computing $h_0(T_{\mathbf{C}}M)$ and $h_1(T_{\mathbf{C}}M)$ (see (5.18) and (5.19)), we have

Corollary 2.1. *If M is 4 dimensional, one has*

$$(2.14) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(4)} = 0.$$

Corollary 2.2. *If M is 12 dimensional, one has*

$$(2.15) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(12)} = -2^{14} \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(12)}.$$

Corollary 2.3. *If M is 20 dimensional, one has*

$$(2.16) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(20)} \\ = 2^{14} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 28 \widehat{A}(TM, \nabla^{TM}) \right\}^{(20)}.$$

We also have,

Theorem 2.2. *For $8k + 4$ dimensional smooth Riemannian manifold M , the following identity holds,*

$$(2.17) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 55 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) + 768 \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\ = 2^{25} \left[\sum_{r=0}^{k-2} (k-r)(k-r-1) 2^{6(k-r-2)} h_r(T_{\mathbf{C}}M) \right],$$

where each $h_r(T_{\mathbf{C}}M) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbf{C}}M)) \right\}^{(8k+4)}$, $0 \leq r \leq k$, and each $b_r(T_{\mathbf{C}}M)$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M)$, $0 \leq j \leq r$. The right hand side is understood as 0 when $k < 2$.

In Theorem 2.2, putting $k = 0, 1, 2, 3$, by computing $h_0(T_{\mathbf{C}}M)$ and $h_1(T_{\mathbf{C}}M)$ (see (5.18) and (5.19)), we have

Corollary 2.4. *If M is 4 dimensional, one has*

$$(2.18) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 55 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) + 768 \widehat{L}(TM, \nabla^{TM}) \right\}^{(4)} = 0.$$

Equivalently, in view of Corollary 2.1, one has

$$(2.19) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 112 \widehat{L}(TM, \nabla^{TM}) \right\}^{(4)} = 0.$$

Corollary 2.5. *If M is 12 dimensional, one has*

$$(2.20) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 55 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) + 768 \widehat{L}(TM, \nabla^{TM}) \right\}^{(12)} = 0.$$

Corollary 2.6. *If M is 20 dimensional, one has*

$$(2.21) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 55 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) + 768 \widehat{L}(TM, \nabla^{TM}) \right\}^{(20)} \\ = -2^{26} \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(20)}.$$

Corollary 2.7. *If M is 28 dimensional, one has*

$$(2.22) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 55 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) + 768 \widehat{L}(TM, \nabla^{TM}) \right\}^{(28)} \\ = 2^{26} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 52 \widehat{A}(TM, \nabla^{TM}) \right\}^{(28)}.$$

For $8k$ dimensional manifolds, we have the following twisted cancellation formulas.

Theorem 2.3. *For $8k$ dimensional smooth Riemannian manifold M , the following identity holds,*

$$(2.23) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8k)} = 2^{11} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} z_r(T_{\mathbf{C}}M) \right],$$

where each $z_r(T_{\mathbf{C}}M) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(d_r(T_{\mathbf{C}}M)) \right\}^{(8k)}$, $0 \leq r \leq k$, and each $d_r(T_{\mathbf{C}}M)$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M)$, $0 \leq j \leq r$.

In Theorem 2.3, putting $k = 1, 2$, by computing $z_0(T_{\mathbf{C}}M)$ and $z_1(T_{\mathbf{C}}M)$ (see (5.32) and (5.33)), we have

Corollary 2.8. *If M is 8 dimensional, one has*

$$(2.24) \quad \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8)} = 2048 \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(8)}.$$

Corollary 2.9. *If M is 16 dimensional, one has*

$$(2.25) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(16)} \\ = -2048 \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 48 \widehat{A}(TM, \nabla^{TM}) \right\}^{(16)}.$$

Theorem 2.4. *For $8k$ dimensional smooth Riemannian manifold M , the following identity holds,*

$$(2.26) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 23 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8k)} \\ = 2^{22} \left[\sum_{r=0}^{k-2} (k-r)(k-r-1) 2^{6(k-r-2)} z_r(T_{\mathbf{C}}M) \right],$$

where each $z_r(T_{\mathbf{C}}M) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(d_r(T_{\mathbf{C}}M)) \right\}^{(8k)}$, $0 \leq r \leq k$, and each $d_r(T_{\mathbf{C}}M)$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M)$, $0 \leq j \leq r$. The right hand side is understood as 0 when $k < 2$.

In Theorem 2.4, putting $k = 1, 2, 3$, by computing $z_0(T_{\mathbf{C}}M)$ and $z_1(T_{\mathbf{C}}M)$ (see (5.32) and (5.33)), we have

Corollary 2.10. *If M is 8 dimensional, one has*

$$(2.27) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 23 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8)} = 0.$$

Equivalently, in view of Corollary 2.8, one has

$$(2.28) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) \right\}^{(8)} = \left\{ 23 \cdot 2048 \widehat{A}(TM, \nabla^{TM}) \right\}^{(8)}.$$

Corollary 2.11. *If M is 16 dimensional, one has*

$$(2.29) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 23 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(16)} \\ = 2^{23} \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(16)}.$$

Corollary 2.12. *If M is 24 dimensional, one has*

$$(2.30) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 23 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(24)} \\ = -2^{23} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 72 \widehat{A}(TM, \nabla^{TM}) \right\}^{(24)}.$$

If we include the extra complex line bundle ξ into our picture, we have the following.

Theorem 2.5. *For $8k + 4$ dimensional smooth Riemannian manifold M , the following identity holds,*

$$(2.31) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM}) [\text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - \sinh^2\left(\frac{\epsilon}{2}\right) \text{ch}(2\xi_{\mathbf{C}} \oplus \mathbf{C}^8) - 16]}{\cosh^2\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+4)} \\ = 2^{14} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right],$$

where each $h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{\epsilon}{2}\right) \right\}^{(8k+4)}$, $0 \leq r \leq k$, and each $b_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$, $0 \leq j \leq r$. The right hand is understood as 0 when $k < 1$.

For $8k$ dimensional case, we have the following.

Theorem 2.6. *For $8k$ dimensional smooth Riemannian manifold M , the following identity holds,*

$$(2.32) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM}) [\text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - \sinh^2\left(\frac{\epsilon}{2}\right) \text{ch}(2\xi_{\mathbf{C}} \oplus \mathbf{C}^8)]}{\cosh^2\left(\frac{\epsilon}{2}\right)} \right\}^{(8k)} \\ = 2^{11} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right],$$

where each $z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(d_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{\epsilon}{2}\right) \right\}^{(8k)}$, $0 \leq r \leq k$, and each $d_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$ is a canonical integral linear combination of $B_j(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$, $0 \leq j \leq r$.

3. SPIN MANIFOLDS AND DIVISIBILITIES OF TWISTED SIGNATURES

Let M be a closed oriented differential manifold and V be a complex vector bundle over M . Let $\text{Sig}(M, V) \triangleq \text{Ind}(d_s \otimes V)$ denote the twisted signature [7]. Let $\text{Sig}(M, T)$ and $\text{Sig}(M, T \otimes T)$ denote $\text{Sig}(M, T_{\mathbf{C}}M)$ and $\text{Sig}(M, T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M)$ respectively. In this section, we apply Theorem 2.1, 2.2 and Theorem 2.3, 2.4 to $8k + 4$ and $8k$ dimensional closed spin manifolds respectively to obtain divisibility results for the twisted signatures $\text{Sig}(M, T)$ and $\text{Sig}(M, T \otimes T)$. We also show that our divisibilities are best possible.

3.1. $8k + 4$ dimensional case. According to the generalized Hirzebruch signature formula [7, 11], when M is an $8k + 4$ dimensional closed spin manifold, integrating both sides of (2.13) against the fundamental class $[M]$, we have

$$(3.1) \quad \text{Sig}(M, T) - 16 \text{Sig}(M) = 2^{14} \left\langle \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M) \right], [M] \right\rangle.$$

According to the Atiyah-Hirzebruch divisibility [3], we have

Theorem 3.1. *If M is an $8k + 4$ dimensional closed spin manifold, then $(\text{Sig}(M, T) - 16 \text{Sig}(M))$ is divisible by 2^{15} .*

Then according to Theorem 3.1 and the Ochanine divisibility [19] that the signature of $8k + 4$ dimensional closed spin manifolds is divisible by 16, we see that our twisted anomaly cancellation formula (2.13) actually implies the Hirzebruch divisibility:

Theorem 3.2 (Hirzebruch, [12]). *If M is an $8k + 4$ dimensional closed spin manifold, then the twisted signature $\text{Sig}(M, T)$ is divisible by 256.*

Moreover, we are able to show that the Hirzebruch divisibility is best possible.

Proposition 3.1. *256 is the best possible divisibility of the twisted signature $\text{Sig}(M, T)$ for $8k + 4$ dimensional spin manifolds.*

To prove Proposition 3.1, we need the following lemmas.

Lemma 3.1. *Let M_1 and M_2 be two closed oriented smooth manifolds, then one has*

$$(3.2) \quad \text{Sig}(M_1 \times M_2, T) = \text{Sig}(M_1)\text{Sig}(M_2, T) + \text{Sig}(M_2)\text{Sig}(M_1, T),$$

and

$$(3.3) \quad \begin{aligned} & \text{Sig}(M_1 \times M_2, T \otimes T) \\ &= \text{Sig}(M_1)\text{Sig}(M_2, T \otimes T) + 2\text{Sig}(M_1, T)\text{Sig}(M_2, T) + \text{Sig}(M_2)\text{Sig}(M_1, T \otimes T). \end{aligned}$$

Proof. Let $p_1 : M_1 \times M_2 \rightarrow M_1, p_2 : M_1 \times M_2 \rightarrow M_2$ be the two projections. It's not hard to see that

$$\begin{aligned} & \int_{M_1 \times M_2} \widehat{L}(M_1 \times M_2) \text{ch}(T_{\mathbf{C}}(M_1 \times M_2)) \\ &= \int_{M_1 \times M_2} \widehat{L}(p_1^*(TM_1)) \widehat{L}(p_2^*(TM_2)) (\text{ch}(p_1^*(T_{\mathbf{C}}M_1)) + \text{ch}(p_2^*(T_{\mathbf{C}}M_2))) \\ &= \int_{M_1 \times M_2} p_1^* \left(\widehat{L}(M_1) \right) p_2^* \left(\widehat{L}(M_2) \text{ch}(T_{\mathbf{C}}M_2) \right) + p_2^* \left(\widehat{L}(M_2) \right) p_1^* \left(\widehat{L}(M_1) \text{ch}(T_{\mathbf{C}}M_1) \right) \\ &= \int_{M_1} \widehat{L}(M_1) \int_{M_2} \widehat{L}(M_2) \text{ch}(T_{\mathbf{C}}M_2) + \int_{M_2} \widehat{L}(M_2) \int_{M_1} \widehat{L}(M_1) \text{ch}(T_{\mathbf{C}}M_1). \end{aligned}$$

Thus we have

$$\text{Sig}(M_1 \times M_2, T) = \text{Sig}(M_1)\text{Sig}(M_2, T) + \text{Sig}(M_2)\text{Sig}(M_1, T).$$

By similar computations, it's not hard to prove (3.3). \square

Lemma 3.2. *Let \mathbf{HP}^2 be the quaternionic projective plane. Then for the $8n$ dimensional manifold $(\mathbf{HP}^2)^n$, the n -fold product of \mathbf{HP}^2 , one has $\text{Sig}((\mathbf{HP}^2)^n) = 1$, $\text{Sig}((\mathbf{HP}^2)^n, T) = 0$ and $\text{Sig}((\mathbf{HP}^2)^n, T \otimes T) = 0$, where n is a positive integer.*

Proof. Let \mathbf{HP}^n be the quaternionic projective space and u be the generator of $H^4(\mathbf{HP}^n, \mathbf{Z})$. A theorem of Hirzebruch ([10]) says the total Pontrjagin class of $T\mathbf{HP}^n$ is the following

$$(3.4) \quad p(\mathbf{HP}^n) = (1 + u)^{2n+2}(1 + 4u)^{-1} = (1 + u)^{2n+2}(1 - 4u + 16u^2 + \dots).$$

In particular, for \mathbf{HP}^2 , we have $p_1(\mathbf{HP}^2) = 2u, p_2(\mathbf{HP}^2) = 7u^2$.

By direct computations, $\text{Sig}(\mathbf{HP}^2) = \langle -\frac{1}{45}p_1^2 + \frac{7}{45}p_2, [\mathbf{HP}^2] \rangle = 1$. Thus by the multiplicity of the signature, we have $\text{Sig}((\mathbf{HP}^2)^n) = 1$, for $n \in \mathbf{Z}^+$.

Also by direct computations, we have $\widehat{A}(\mathbf{HP}^2) = \langle \frac{1}{16}(\frac{7}{360}p_1^2 - \frac{1}{90}p_2), [\mathbf{HP}^2] \rangle = 0$. Thus by Corollary 2.8, we have $\text{Sig}(\mathbf{HP}^2, T) = 0$. Keeping applying (3.2), we obtain that $\text{Sig}((\mathbf{HP}^2)^n, T) = 0$, for $n \in \mathbf{Z}^+$.

By Corollary 2.10, $\text{Sig}(\mathbf{HP}^2, T \otimes T) = 23 \text{Sig}(\mathbf{HP}^2, T) = 0$. Keeping applying (3.3), we obtain that $\text{Sig}((\mathbf{HP}^2)^n, T \otimes T) = 0$, for $n \in \mathbf{Z}^+$. \square

Now we can prove Proposition 3.1 as follows.

Proof. Let K be a K3-surface. It is well known that $\text{Sig}(K)$ is -16 . Then by Corollary 2.1, $\text{Sig}(K, T) = 16 \text{Sig}(K) = -256$.

Applying (3.2) and Lemma 3.2 to the $8k + 4$ dimensional spin manifold $K \times (\mathbf{HP}^2)^k$, we obtain that

$$\begin{aligned} & \text{Sig}(K \times (\mathbf{HP}^2)^k, T) \\ &= \text{Sig}(K) \text{Sig}((\mathbf{HP}^2)^k, T) + \text{Sig}((\mathbf{HP}^2)^k) \text{Sig}(K, T) \\ &= \text{Sig}(K, T) \\ &= -256. \end{aligned}$$

This proves Proposition 3.1. \square

Our twisted anomaly cancellation formula (2.17) implies the divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$. According to the generalized Hirzebruch signature formula [7, 11], when M is an $8k + 4$ dimensional closed spin manifold, integrating both sides of (2.17) against the fundamental class $[M]$, we have

$$\begin{aligned} & \text{Sig}(M, T \otimes T) - 55 \text{Sig}(M, T) + 768 \text{Sig}(M) \\ (3.5) \quad &= 2^{25} \langle [\sum_{r=0}^{k-2} (k-r)(k-r-1) 2^{6(k-r-2)} h_r(T_{\mathbf{C}} M)], [M] \rangle. \end{aligned}$$

In particular, by (2.19), in dimension 4, we have

$$(3.6) \quad \text{Sig}(M, T \otimes T) - 112 \text{Sig}(M) = 0.$$

By the Atiyah-Hirzebruch divisibility [3], (3.5) shows $\text{Sig}(M, T \otimes T) - 55 \text{Sig}(M, T) + 768 \text{Sig}(M)$ is divisible by 2^{26} . Therefore, according to Theorem 3.2 and the Ochanine divisibility [19], we obtain

Theorem 3.3. *Let M be an $8k+4$ dimensional closed spin manifold, when $\dim M = 4$, the twisted signature $\text{Sig}(M, T \otimes T)$ is divisible by $256 \cdot 7$; when $\dim M = 8k+4$, $k \geq 1$, the the twisted signature $\text{Sig}(M, T \otimes T)$ is divisible by 256.*

Moreover we are also able to show that these divisibilities of $\text{Sig}(M, T \otimes T)$ are best possible.

Proposition 3.2. *$256 \cdot 7$ is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for 4 dimensional spin manifolds; 256 is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for $8k+4$ dimensional spin manifolds, where $k \geq 1$.*

Proof. Let K be a K3-surface. By (3.6), one has $\text{Sig}(K, T \otimes T) = 112 \text{Sig}(K) = -256 \cdot 7$. This shows $256 \cdot 7$ is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for 4 dimensional spin manifolds.

Let B^8 be such a Bott manifold, which is 8 dimensional, spin with $\hat{A}(B^8) = 1$ and $\text{Sig}(B^8) = 0$ [15]. By Corollary 2.8, $\text{Sig}(B^8, T) = 2048 \hat{A}(B^8) = 2048$. Then by (2.27), $\text{Sig}(B^8, T \otimes T) = 23 \text{Sig}(B^8, T) = 23 \cdot 2048$. Therefore, by (3.3),

$$\begin{aligned} & \text{Sig}(K \times B^8, T \otimes T) \\ &= \text{Sig}(K) \text{Sig}(B^8, T \otimes T) + 2 \text{Sig}(K, T) \text{Sig}(B^8, T) + \text{Sig}(B^8) \text{Sig}(K, T \otimes T) \\ &= -16 \cdot 23 \cdot 2048 + 2 \cdot (-256) \cdot 2048 \\ &= -55 \cdot 2^{15}. \end{aligned}$$

Applying (3.3) and Lemma 3.2 to the $8k + 4$ dimensional, $k \geq 1$, spin manifold $(\mathbf{HP}^2)^{k-1} \times K \times B^8$, we have

$$\begin{aligned} & \text{Sig}((\mathbf{HP}^2)^{k-1} \times K \times B^8, T \otimes T) \\ &= \text{Sig}(K \times B^8) \text{Sig}((\mathbf{HP}^2)^{k-1}, T \otimes T) + 2 \text{Sig}(K \times B^8, T) \text{Sig}((\mathbf{HP}^2)^{k-1}, T) \\ & \quad + \text{Sig}((\mathbf{HP}^2)^{k-1}) \text{Sig}(K \times B^8, T \otimes T) \\ &= \text{Sig}(K \times B^8, T \otimes T) \\ &= -55 \cdot 2^{15}. \end{aligned}$$

Applying (3.3) and Lemma 3.2 to the $8k + 4$ dimensional, $k \geq 1$, spin manifold $(\mathbf{HP}^2)^k \times K$, we have

$$\begin{aligned} & \text{Sig}((\mathbf{HP}^2)^k \times K, T \otimes T) \\ &= \text{Sig}(K) \text{Sig}((\mathbf{HP}^2)^k, T \otimes T) + 2 \text{Sig}(K, T) \text{Sig}((\mathbf{HP}^2)^k, T) \\ & \quad + \text{Sig}((\mathbf{HP}^2)^k) \text{Sig}(K, T \otimes T) \\ &= \text{Sig}(K, T \otimes T) \\ &= -256 \cdot 7. \end{aligned}$$

Since the maximal common denominator of $-55 \cdot 2^{15}$ and $-256 \cdot 7$ is 256, we see that 256 is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for $8k + 4$ dimensional spin manifolds, where $k \geq 1$. \square

3.2. $8k$ dimensional case. According to the generalized Hirzebruch signature formula [7], when M is an $8k$ dimensional closed spin manifold, integrating both sides of (2.23) against the fundamental class $[M]$, we have

$$(3.7) \quad \text{Sig}(M, T) = 2048 \left(\left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} z_r(T_{\mathbf{C}} M) \right], [M] \right).$$

Then according to the Atiyah-Singer index theorem, our anomaly cancellation formula (2.23) actually implies the Hirzebruch divisibility:

Theorem 3.4 (Hirzebruch, [12]). *If M is an $8k$ dimensional closed spin manifold, then $\text{Sig}(M, T)$ is divisible by 2048.*

Remark 3.1. *This divisibility looks astonishing since on $8k$ dimensional closed spin manifold, we can say nothing about the divisibility on the signature while this twisted signature has so high divisibility.*

Moreover, we are able to show that the Hirzebruch divisibility is best possible.

Proposition 3.3. *2048 is the best possible divisibility for the tangent twisted signature of $8k$ dimensional spin manifolds.*

Proof. Let B^8 be a Bott manifold as in the proof of Proposition 3.2. Applying Lemma 3.1 and Lemma 3.2 to the $8k$ dimensional spin manifold $B^8 \times (\mathbf{H}P^2)^{k-1}$, one has

$$\begin{aligned} & \text{Sig}(B^8 \times (\mathbf{H}P^2)^{k-1}, T) \\ &= \text{Sig}(B^8) \text{Sig}((\mathbf{H}P^2)^{k-1}, T) + \text{Sig}((\mathbf{H}P^2)^{k-1}) \text{Sig}(B^8, T) \\ &= \text{Sig}(B^8, T) \\ &= 2048. \end{aligned}$$

This proves Proposition 3.3. \square

Our twisted anomaly cancellation formula (2.26) implies the divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$. According to the generalized Hirzebruch signature formula [7, 11], when M is an $8k$ dimensional closed spin manifold, integrating both sides of (2.26) against the fundamental class $[M]$, we have

$$\begin{aligned} & \text{Sig}(M, T \otimes T) - 23 \text{Sig}(M, T) \\ (3.8) \quad &= 2^{22} \langle [\sum_{r=0}^{k-2} (k-r)(k-r-1) 2^{6(k-r-2)} z_r(T_{\mathbf{C}}M)], [M] \rangle. \end{aligned}$$

In particular, by (2.28), in dimension 8, we have

$$(3.9) \quad \text{Sig}(M, T \otimes T) - 23 \cdot 2048 \hat{A}(M) = 0.$$

By the Atiyah-Singer index theorem, $\text{Sig}(M, T \otimes T) - 23 \text{Sig}(M, T)$ is divisible by 2^{22} . Therefore, according to Theorem 3.4, we obtain

Theorem 3.5. *Let M be an $8k$ dimensional closed spin manifold, when $\dim M = 8$, the twisted signature $\text{Sig}(M, T \otimes T)$ is divisible by $2048 \cdot 23$; when $\dim M = 8k, k \geq 2$, the the twisted signature $\text{Sig}(M, T \otimes T)$ is divisible by 2048.*

Moreover, we are also able to show that these divisibilities of $\text{Sig}(M, T \otimes T)$ are best possible.

Proposition 3.4. *$2048 \cdot 23$ is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for 8 dimensional spin manifolds; 2048 is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for $8k$ dimensional spin manifolds, where $k \geq 2$.*

Proof. Let B^8 be a Bott manifold as in the proof of Proposition 3.2. $\text{Sig}(B^8, T \otimes T) = 2048 \cdot 23 \hat{A}(B^8) = 2048 \cdot 23$ shows that $2048 \cdot 23$ is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for 8 dimensional spin manifolds.

By (3.3),

$$\begin{aligned} & \text{Sig}(B^8 \times B^8, T \otimes T) \\ &= \text{Sig}(B^8) \text{Sig}(B^8, T \otimes T) + 2 \text{Sig}(B^8, T) \text{Sig}(B^8, T) + \text{Sig}(B^8) \text{Sig}(B^8, T \otimes T) \\ &= 2 \text{Sig}(B^8, T) \text{Sig}(B^8, T) \\ &= 2^{23}. \end{aligned}$$

Applying (3.3) and Lemma 3.2 to the $8k$ dimensional, $k \geq 2$, spin manifold $(\mathbf{H}P^2)^{k-2} \times B^8 \times B^8$, we have

$$\begin{aligned}
& \text{Sig}((\mathbf{H}P^2)^{k-2} \times B^8 \times B^8, T \otimes T) \\
&= \text{Sig}(B^8 \times B^8) \text{Sig}((\mathbf{H}P^2)^{k-2}, T \otimes T) + 2 \text{Sig}(B^8 \times B^8, T) \text{Sig}((\mathbf{H}P^2)^{k-2}, T) \\
&\quad + \text{Sig}((\mathbf{H}P^2)^{k-2}) \text{Sig}(B^8 \times B^8, T \otimes T) \\
&= \text{Sig}(B^8 \times B^8, T \otimes T) \\
&= 2^{23}.
\end{aligned}$$

Applying (3.3) and Lemma 3.2 to the $8k$ dimensional, $k \geq 2$, spin manifold $(\mathbf{H}P^2)^{k-1} \times B^8$, we have

$$\begin{aligned}
& \text{Sig}((\mathbf{H}P^2)^{k-1} \times B^8, T \otimes T) \\
&= \text{Sig}(B^8) \text{Sig}((\mathbf{H}P^2)^{k-1}, T \otimes T) + 2 \text{Sig}(B^8, T) \text{Sig}((\mathbf{H}P^2)^{k-1}, T) \\
&\quad + \text{Sig}((\mathbf{H}P^2)^{k-1}) \text{Sig}(B^8, T \otimes T) \\
&= \text{Sig}(B^8, T \otimes T) \\
&= 2048 \cdot 23.
\end{aligned}$$

Since the maximal common denominator of 2^{23} and $2048 \cdot 23$ is 2048, we see that 2048 is the best possible divisibility of the twisted signature $\text{Sig}(M, T \otimes T)$ for $8k$ dimensional spin manifolds, where $k \geq 2$. \square

4. Spin^c MANIFOLDS AND TWISTED ROKHLIN CONGRUENCES FOR CHARACTERISTIC NUMBERS

In this section, we apply Theorem 2.5 and Theorem 2.6 to $8k+4$ and $8k$ dimensional spin^c manifolds respectively to obtain some congruence results. In particular, for $8k+4$ dimensional spin^c manifolds, we establish twisted Rokhlin congruence formulas.

4.1. $8k+4$ dimensional case. Let M be an $8k+4$ dimensional closed spin^c manifold and (ξ, ∇^ξ) be a real oriented Euclidean plane bundle, or equivalently a complex line bundle, over M such that $w_2(TM) \equiv [e(\xi, \nabla^\xi)]$ in $H^2(M, \mathbf{Z}_2)$. Let B be an oriented $8k+2$ dimensional submanifold of M such that $[B] \in H_{8k+2}(M, \mathbf{Z})$ is dual to $[e(\xi, \nabla^\xi)]$. Let $B \bullet B$ denote the self-intersection of B in M and N be the normal bundle to $B \bullet B$ in M . Applying the Poincaré duality, we have

$$\begin{aligned}
(4.1) \quad \int_M \frac{\widehat{L}(TM)}{\cosh^2(\frac{c}{2})} &= \int_M \widehat{L}(TM) - \int_M \widehat{L}(TM) \frac{\sinh^2(\frac{c}{2})}{\cosh^2(\frac{c}{2})} \\
&= \int_M \widehat{L}(TM) - \int_{B \bullet B} \widehat{L}(B \bullet B),
\end{aligned}$$

$$\begin{aligned}
(4.2) \quad & \int_M \frac{\widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M)}{\cosh^2(\frac{c}{2})} \\
&= \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) - \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) \frac{\sinh^2(\frac{c}{2})}{\cosh^2(\frac{c}{2})} \\
&= \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) - \int_{B \bullet B} \widehat{L}(B \bullet B) \text{ch}(T_{\mathbf{C}}(B \bullet B) \oplus N_{\mathbf{C}}),
\end{aligned}$$

and

$$(4.3) \quad \int_M \widehat{L}(TM) \text{ch}(2\xi_{\mathbf{C}} \oplus \mathbf{C}^8) \frac{\sinh^2(\frac{c}{2})}{\cosh^2(\frac{c}{2})} = \int_{B \bullet B} \widehat{L}(B \bullet B) \text{ch}(N_{\mathbf{C}} \oplus \mathbf{C}^8).$$

Therefore, by (2.31), (4.1) to (4.3), we have

$$(4.4) \quad \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) - \int_{B \bullet B} \widehat{L}(B \bullet B) \text{ch}(T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \\ - 16 \left(\int_M \widehat{L}(TM) - \int_{B \bullet B} \widehat{L}(B \bullet B) \right) = 2^{14} \int_M \sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

Thus one has

$$(4.5) \quad \frac{1}{128} \left\{ \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) - \int_{B \bullet B} \widehat{L}(B \bullet B) \text{ch}(T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \right\} \\ = \frac{\int_M \widehat{L}(TM) - \int_{B \bullet B} \widehat{L}(B \bullet B)}{8} + 2^7 \int_M \sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}),$$

and

$$(4.6) \quad \frac{1}{2^{14}} \left\{ \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) - \int_{B \bullet B} \widehat{L}(B \bullet B) \text{ch}(T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \right. \\ \left. - 16 \left(\int_M \widehat{L}(TM) - \int_{B \bullet B} \widehat{L}(B \bullet B) \right) \right\} \\ = \int_M \sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \\ = \int_M \sum_{r=0}^{k-2} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) + \int_M h_{k-1}(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \\ = \int_M \sum_{r=0}^{k-2} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \\ + \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_{k-1}(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{c}{2}\right).$$

From [9, Theorem 3.2] (Theorem 4.2 below), Theorem 2.5, (4.5), (4.6) and the Atiyah-Singer index theorem for spin^c manifolds, we obtain that

Theorem 4.1. *If M is an $8k+4$ dimensional closed spin^c manifold, ξ is a complex line bundle over M such that $c_1(\xi) \equiv w_2(TM) \in H^2(M, \mathbf{Z}_2)$ and B is an oriented $8k+2$ dimensional submanifold of M such that $[B] \in H_{8k+2}(M, \mathbf{Z})$ is dual to $c_1(\xi) \in H^2(M, \mathbf{Z})$, then*

$$\text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8)$$

is divisible by 128 and

$$\text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) - 16(\text{Sig}(M) - \text{Sig}(B \bullet B))$$

is divisible by 2^{14} . Moreover, one has

$$(4.7) \quad \begin{aligned} & \frac{1}{128} \{ \text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \} \\ & \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_k(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{c}{2}\right) \pmod{64}, \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} & \frac{1}{2^{14}} \{ \text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) - 16(\text{Sig}(M) - \text{Sig}(B \bullet B)) \} \\ & \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_{k-1}(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{c}{2}\right) \pmod{128}. \end{aligned}$$

Note that we also have the following results:

Theorem 4.2. (Han – Zhang [9, Theorem 3.2]) *The following congruence formula holds,*

$$(4.9) \quad \begin{aligned} & \frac{\text{Sig}(M) - \text{Sig}(B \bullet B)}{8} \\ & \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_k(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{c}{2}\right) \pmod{64}. \end{aligned}$$

Theorem 4.3. (Liu – Zhang [18, Theorem 4.2], Han – Zhang [9, formula 3.5, 3.6 and 3.14]) *The following congruence formulas hold,*

$$(4.10) \quad \begin{aligned} & \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{c}{2}\right) \\ & \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbf{C}}M + \mathbf{C}^2 - \xi_{\mathbf{C}}, \mathbf{C}^2)) \cosh\left(\frac{c}{2}\right) \pmod{2}, \end{aligned}$$

$0 \leq r \leq k$, and

$$(4.11) \quad \begin{aligned} & \frac{\text{Sig}(M) - \text{Sig}(B \bullet B)}{8} \\ & \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_k(T_{\mathbf{C}}M + \mathbf{C}^2 - \xi_{\mathbf{C}}, \mathbf{C}^2)) \cosh\left(\frac{c}{2}\right) \pmod{2}. \end{aligned}$$

Let E be a real vector bundle over M and $i : B \hookrightarrow M$ denote the canonical embedding of B in M . Then we have ,

Theorem 4.4. (Zhang, [20]) *The following identity holds,*

$$(4.12) \quad \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(E \otimes \mathbf{C}) \cosh\left(\frac{c}{2}\right) \equiv \text{ind}_2(i^*E) \pmod{2},$$

where $\text{ind}_2(i^*E)$ is the mod 2 index in the sense of Atiyah and Singer.

Note that Theorem 4.4 only holds for $8k + 4$ dimensional spin^c manifolds.

Combining Theorem 4.1, Theorem 4.3 and 4.4, we obtain that

Corollary 4.1. *The following congruence formulas hold,*

$$(4.13) \quad \begin{aligned} & \frac{1}{128} \{ \text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \} \\ & \equiv \text{ind}_2(b_k(TB + \mathbf{R}^2, \mathbf{R}^2)) \pmod{2}, \end{aligned}$$

and

$$(4.14) \quad \frac{1}{2^{14}} \{ \text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) - 16(\text{Sig}(M) - \text{Sig}(B \bullet B)) \} \\ \equiv \text{ind}_2(b_{k-1}(TB + \mathbf{R}^2, \mathbf{R}^2)) \pmod{2}.$$

We can regard (4.13) and (4.14) as twisted Rokhlin congruence formulas for $8k + 4$ dimensional spin^c manifolds.

4.2. $8k$ dimensional case. By similar computations as what we did in the above subsection, for $8k$ dimensional case, from Theorem 2.6, we have

$$(4.15) \quad \int_M \widehat{L}(TM) \text{ch}(T_{\mathbf{C}}M) - \int_{B \bullet B} \widehat{L}(B \bullet B) \text{ch}(T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \\ = 2^{11} \int_M \sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

Theorem 4.5. *If M is an $8k$ dimensional closed spin^c manifold, ξ is a complex line bundle over M such that $c_1(\xi) \equiv w_2(TM) \in H^2(M, \mathbf{Z}_2)$ and B is an oriented $8k - 2$ dimensional submanifold of M such that $[B] \in H_{8k-2}(M, \mathbf{Z})$ is dual to $c_1(\xi) \in H^2(M, \mathbf{Z})$. Let $B \bullet B$ be the self-intersection of B in M and N be the normal bundle of $B \bullet B$ in M , then*

$$\text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8)$$

is divisible by 2048. Moreover, one has

$$(4.16) \quad \frac{1}{2048} \{ \text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8) \} \\ \equiv \int_M z_{k-1}(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \int_M \widehat{A}(TM, \nabla^{TM}) \text{ch}(d_{k-1}(T_{\mathbf{C}}M, \xi_{\mathbf{C}})) \cosh\left(\frac{c}{2}\right) \pmod{128}.$$

Remark 4.1. *It's pretty interesting to note that in $8k$ dimensional spin^c case although we can say nothing about the divisibility of $(\text{Sig}(M) - \text{Sig}(B \bullet B))$, we do have a very high divisibility for the twisted version*

$$\text{Sig}(M, T_{\mathbf{C}}M) - \text{Sig}(B \bullet B, T_{\mathbf{C}}(B \bullet B) \oplus 2N_{\mathbf{C}} \oplus \mathbf{C}^8),$$

which is even much higher than the $8k + 4$ dimensional case.

5. PROOFS OF TWISTED ANOMALY CANCELLATION FORMULAS

We use the modular invariance method developed in [16, 8, 9] to prove Theorem 2.1 to 2.6 in this section.

We first recall some necessary knowledge on theta-functions and modular forms. Then in Section 5.1 we prove Theorem 2.1 to 2.4 together and in Section 5.2 we prove Theorem 2.5 and 2.6 together.

Recall that the four Jacobi theta-functions [5] defined by infinite multiplications are

$$(5.1) \quad \theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j)],$$

$$(5.2) \quad \theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j)],$$

$$(5.3) \quad \theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi\sqrt{-1}v} q^{j-1/2})],$$

$$(5.4) \quad \theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi\sqrt{-1}v} q^{j-1/2})],$$

where $q = e^{2\pi\sqrt{-1}\tau}$, $\tau \in \mathbf{H}$.

They are all holomorphic functions for $(v, \tau) \in \mathbf{C} \times \mathbf{H}$, where \mathbf{C} is the complex plane and \mathbf{H} is the upper half plane.

Let $\theta'(0, \tau) = \frac{\partial}{\partial v} \theta(v, \tau)|_{v=0}$, then the following Jacobi identity relates the four theta-functions garcefully.

Proposition 5.1. (Jacobi identity, [5, Chapter 3]) *The following identity holds,*

$$(5.5) \quad \theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau).$$

Let

$$SL_2(\mathbf{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, ad - bc = 1 \right\}$$

as usual be the famous modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the two generators of $SL_2(\mathbf{Z})$. Their actions on \mathbf{H} are given by

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1.$$

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \middle| c \equiv 0 \pmod{2} \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \middle| b \equiv 0 \pmod{2} \right\}$$

be the two modular subgroup of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST , while the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [5]).

If we act theta-functions by S and T , the following transformation formulas hold (cf. [5]),

$$(5.6) \quad \theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau);$$

$$(5.7) \quad \theta_1(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau);$$

$$(5.8) \quad \theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau);$$

$$(5.9) \quad \theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau).$$

Let Γ be a subgroup of $SL_2(\mathbf{Z})$.

Definition 5.1. *A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbf{H} \cup \{\infty\}$ such that for any*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

the following property holds

$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau),$$

where $\chi : \Gamma \rightarrow \mathbf{C}^$ is a character of Γ and k is called the weight of f .*

If Γ is a modular subgroup, let $\mathcal{M}_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Writing simply $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$, we introduce four explicit modular forms (cf. [16]),

$$\begin{aligned} \delta_1(\tau) &= \frac{1}{8}(\theta_2^4 + \theta_3^4), & \varepsilon_1(\tau) &= \frac{1}{16}\theta_2^4\theta_3^4, \\ \delta_2(\tau) &= -\frac{1}{8}(\theta_1^4 + \theta_3^4), & \varepsilon_2(\tau) &= \frac{1}{16}\theta_1^4\theta_3^4. \end{aligned}$$

They have the following Fourier expansions in $q^{1/2}$:

$$\begin{aligned} \delta_1(\tau) &= \frac{1}{4} + 6q + 6q^2 + \cdots, & \varepsilon_1(\tau) &= \frac{1}{16} - q + 7q^2 + \cdots, \\ \delta_2(\tau) &= -\frac{1}{8} - 3q^{1/2} - 3q + \cdots, & \varepsilon_2(\tau) &= q^{1/2} + 8q + \cdots. \end{aligned}$$

where the “ \cdots ” terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws (cf [14], [16]),

$$(5.10) \quad \delta_2\left(-\frac{1}{\tau}\right) = \tau^2 \delta_1(\tau) \quad , \quad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4 \varepsilon_1(\tau).$$

Lemma 5.1. ([16]) *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, while $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, and moreover $\mathcal{M}_{\mathbf{R}}(\Gamma^0(2)) = \mathbf{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.*

5.1. Proof of Theorem 2.1 to 2.4. Without loss of generality, we will adopt the Chern roots formalism as in [16], in the computations of characteristic forms.

Recall that if $\{w_i\}$ are the fromal Chern roots of a Hermitian vector bundle E carrying a Hermitian ∇^E , then one has the following formula for the Chern character form of the exterior power of E [11],

$$(5.11) \quad \text{ch}(\Lambda_t(E)) = \prod_i (1 + e^{w_i t}).$$

Let's deal with $8k + 4$ dimensional manifolds first.

For $\tau \in \mathbf{H}$ and $q = e^{2\pi\sqrt{-1}\tau}$, set (cf. [16])

$$(5.12) \quad P_1(\tau) = \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch} \left(\Theta_1(T_{\mathbf{C}}M), \nabla^{\Theta_1(T_{\mathbf{C}}M)} \right) \right\}^{(8k+4)},$$

$$(5.13) \quad P_2(\tau) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} \left(\Theta_2(T_{\mathbf{C}}M), \nabla^{\Theta_2(T_{\mathbf{C}}M)} \right) \right\}^{(8k+4)},$$

where $\nabla^{\Theta_i(T_{\mathbf{C}}M)}$, $i = 1, 2$, are the Hermitian connections with $q^{j/2}$ -coefficients on $\Theta_i(T_{\mathbf{C}}M)$ induced from those on the $A_j(T_{\mathbf{C}}M)$'s and $B_j(T_{\mathbf{C}}M)$'s.

Let $\{\pm 2\pi\sqrt{-1}x_j\}$ be the formal Chern roots for $(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M})$. In terms of the theta-functions, we get (cf. [16])

$$(5.14) \quad P_1(\tau) = 2^{4k+2} \left\{ \prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)} \right\}^{(8k+4)},$$

$$(5.15) \quad P_2(\tau) = \left\{ \prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_2(x_j, \tau)}{\theta_2(0, \tau)} \right\}^{(8k+4)}.$$

Applying the transformation laws (5.6) to (5.9) for theta-functions, we see that $P_1(\tau)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$; while $P_2(\tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Moreover, the following identity holds,

$$(5.16) \quad P_1(-1/\tau) = (2\tau)^{4k+2} P_2(\tau).$$

Observe that at any point $x \in M$, up to the volume form determined by the metric on $T_x M$, both $P_i(\tau)$, $i = 1, 2$, can be viewed as a power series of $q^{1/2}$ with real Fourier coefficients. Thus, one can apply Lemma 5.1 to $P_2(\tau)$ to get, at x , that

$$(5.17) \quad P_2(\tau) = h_0(T_{\mathbf{C}}M)(8\delta_2)^{2k+1} + h_1(T_{\mathbf{C}}M)(8\delta_2)^{2k-1}\varepsilon_2 + \cdots + h_k(T_{\mathbf{C}}M)(8\delta_2)\varepsilon_2^k,$$

where each $h_r(T_{\mathbf{C}}M)$, $0 \leq r \leq k$, is a real multiple of the volume form at x .

We can show that each $h_r(T_{\mathbf{C}}M)$, $0 \leq r \leq k$, can be expressed through a canonical integral linear combination of $\left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} \left(B_j(T_{\mathbf{C}}M), \nabla^{B_j(T_{\mathbf{C}}M)} \right) \right\}^{(8k+4)}$, $0 \leq j \leq r$, with coefficients not depending on $x \in M$. As in [16], one can use the induction method to prove this fact easily by comparing the coefficients of $q^{j/2}$, $j \geq 0$, between the two sides of (5.17). For the consideration of the length of this paper, we do not give details here but only write down the explicit expressions for $h_0(T_{\mathbf{C}}M)$ and $h_1(T_{\mathbf{C}}M)$ as follows.

$$(5.18) \quad h_0(T_{\mathbf{C}}M) = - \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(8k+4)},$$

$$(5.19) \quad h_1(T_{\mathbf{C}}M) = \left\{ \widehat{A}(TM, \nabla^{TM}) \left[24(2k+1) - \text{ch} \left(B_1(T_{\mathbf{C}}M), \nabla^{B_1(T_{\mathbf{C}}M)} \right) \right] \right\}^{(8k+4)}.$$

By (5.16) and (5.17), we have

$$\begin{aligned}
(5.20) \quad P_1(\tau) &= 2^{4k+2} \frac{1}{\tau^{4k+2}} P_2(-1/\tau) \\
&= 2^{4k+2} \frac{1}{\tau^{4k+2}} \left[h_0(T_{\mathbf{C}}M) (8\delta_2(-1/\tau))^{2k+1} + h_1(T_{\mathbf{C}}M) (8\delta_2(-1/\tau))^{2k-1} \varepsilon_2(-1/\tau) + \cdots \right. \\
&\quad \left. + h_k(T_{\mathbf{C}}M) (8\delta_2(-1/\tau)) (\varepsilon_2(-1/\tau))^k \right] \\
&= 2^{4k+2} \left[h_0(T_{\mathbf{C}}M) (8\delta_1)^{2k+1} + h_1(T_{\mathbf{C}}M) (8\delta_1)^{2k-1} \varepsilon_1 + \cdots + h_k(T_{\mathbf{C}}M) (8\delta_1) \varepsilon_1^k \right].
\end{aligned}$$

Expanding $\Theta_1(T_{\mathbf{C}}M)$ explicitly, by (2.8) we have

$$\begin{aligned}
(5.21) \quad \Theta_1(T_{\mathbf{C}}M) &= \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbf{C}}M) \Lambda_{-q^n}(\mathbf{C}^{8k+4}) \\
&\quad \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(T_{\mathbf{C}}M) S_{-q^m}(\mathbf{C}^{8k+4}) \\
&= (1 + (T_{\mathbf{C}}M)q + (S^2 T_{\mathbf{C}}M)q^2 + \cdots)(1 + (T_{\mathbf{C}}M)q^2 + \cdots) \\
&\quad (1 - \mathbf{C}^{8k+4}q + (\Lambda^2 \mathbf{C}^{8k+4})q^2 + \cdots)(1 - \mathbf{C}^{8k+4}q^2 + \cdots) \\
&\quad (1 + (T_{\mathbf{C}}M)q + (\Lambda^2 T_{\mathbf{C}}M)q^2 + \cdots)(1 + (T_{\mathbf{C}}M)q^2 + \cdots) \\
&\quad (1 - \mathbf{C}^{8k+4}q + (S^2 \mathbf{C}^{8k+4})q^2 + \cdots)(1 - \mathbf{C}^{8k+4}q^2 + \cdots) \\
&= (1 + 2(T_{\mathbf{C}}M)q + (T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + S^2 T_{\mathbf{C}}M + \Lambda^2 T_{\mathbf{C}}M)q^2 + \cdots) \\
&\quad (1 + 2(T_{\mathbf{C}}M)q^2 + \cdots) \\
&\quad (1 - 2\mathbf{C}^{8k+4}q + (\mathbf{C}^{8k+4} \otimes \mathbf{C}^{8k+4} + S^2 \mathbf{C}^{8k+4} + \Lambda^2 \mathbf{C}^{8k+4})q^2 + \cdots) \\
&\quad (1 - 2\mathbf{C}^{8k+4}q^2 + \cdots) \\
&= (1 + 2(T_{\mathbf{C}}M)q + 2(T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M)q^2 + \cdots) \\
&\quad (1 - 2\mathbf{C}^{8k+4}q + 2(\mathbf{C}^{8k+4} \otimes \mathbf{C}^{8k+4} - \mathbf{C}^{8k+4})q^2 + \cdots) \\
&= 1 + 2(T_{\mathbf{C}}M - \mathbf{C}^{8k+4})q \\
&\quad + 2[-(16k+7)T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + (8k+4)(8k+3)]q^2 + \cdots,
\end{aligned}$$

where the “ \cdots ” are the terms involving q^j 's with $j \geq 3$.

Note that

$$\begin{aligned}
(5.22) \quad & (8\delta_1)^{2k+1-2r} \varepsilon_1^r \\
&= (2 + 48q + 48q^2 \cdots)^{2k+1-2r} \left(\frac{1}{16} - q + 7q^2 \cdots \right)^r \\
&= 2^{2k+1-6r} [1 + 24(2k+1-2r)q + 24(2k+1-2r)(24k-24r+1)q^2 \cdots] \\
&\quad [1 - 16rq + 16(8r^2 - r)q^2 \cdots] \\
&= 2^{2k+1-6r} [1 + (48k+24-64r)q + \\
&\quad (1152k^2 - 3072kr + 2048r^2 + 624k - 1024r + 24)q^2 + \cdots].
\end{aligned}$$

Therefore, by (5.12), (5.21) and (5.22), setting $q = 0$ in (5.20), we get the result of Liu ([16])

$$(5.23) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M).$$

On the other hand, by (5.12), (5.21) and (5.22), comparing the coefficients of q in (5.20), we have

$$(5.24) \quad \begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(2(T_{\mathbf{C}}M) - 2(8k+4)) \right\}^{(8k+4)} \\ &= 8 \sum_{r=0}^k h_r(T_{\mathbf{C}}M) 2^{6k-6r} (48k + 24 - 64r). \end{aligned}$$

Thus

$$(5.25) \quad \begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}((T_{\mathbf{C}}M) - (8k+4)) \right\}^{(8k+4)} \\ &= 4 \sum_{r=0}^k h_r(T_{\mathbf{C}}M) 2^{6k-6r} (48k + 24 - 64r) \\ &= 12 \left(8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M) \right) - 8k \left(8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M) \right) \\ & \quad + \sum_{r=0}^k h_r(T_{\mathbf{C}}M) 2^{6k-6r} (256k - 256r) \\ &= 12 \left(8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M) \right) - 8k \left(8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M) \right) \\ & \quad + 256 \cdot 2^6 \sum_{r=0}^{k-1} (k-r) h_r(T_{\mathbf{C}}M) 2^{6k-6r-6}. \end{aligned}$$

Combining (5.23) and (5.25), one has

$$(5.26) \quad \begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\ &= 2^8 \left[\sum_{r=0}^k (k-r) 2^{6(k-r)} h_r(T_{\mathbf{C}}M) \right] \\ &= 2^{14} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M) \right], \end{aligned}$$

which is just (2.13).

Furthermore, by (5.12), (5.21) and (5.22), comparing the coefficients of q^2 in (5.20), we have

$$\begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(2[-(16k+7)T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + (8k+4)(8k+3)]) \right\}^{(8k+4)} \\ &= 8 \sum_{r=0}^k h_r(T_{\mathbf{C}}M) 2^{6k-6r} (1152k^2 - 3072kr + 2048r^2 + 624k - 1024r + 24). \end{aligned}$$

Thus, combining (5.23) and (5.26), we have

$$\begin{aligned}
& \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}([-(16k+7)T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + (8k+4)(8k+3)]) \right\}^{(8k+4)} \\
&= 32 \sum_{r=0}^k h_r(T_{\mathbf{C}}M) 2^{6k-6r} (144k^2 - 384kr + 256r^2 + 78k - 128r + 3) \\
&= -16k \cdot 2^8 \left[\sum_{r=0}^k (k-r) 2^{6(k-r)} h_r(T_{\mathbf{C}}M) \right] + 48 \cdot 2^8 \left[\sum_{r=0}^k (k-r) 2^{6(k-r)} h_r(T_{\mathbf{C}}M) \right] \\
&\quad + (64k^2 - 200k + 12) \cdot 8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M) \\
&\quad + 2^{13} \sum_{r=0}^k (k-r)(k-r-1) 2^{6(k-r)} h_r(T_{\mathbf{C}}M) \\
&= -16k \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\
&\quad + 48 \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \text{ch}(T_{\mathbf{C}}M) - 16 \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\
&\quad + (64k^2 - 200k + 12) \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \right\}^{(8k+4)} \\
&\quad + 2^{13} \sum_{r=0}^k (k-r)(k-r-1) 2^{6(k-r)} h_r(T_{\mathbf{C}}M).
\end{aligned}$$

Therefore by above computations, we have

$$\begin{aligned}
& \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 55 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) + 768 \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\
&= 2^{13} \sum_{r=0}^k (k-r)(k-r-1) 2^{6(k-r)} h_r(T_{\mathbf{C}}M) \\
&= 2^{25} \sum_{r=0}^{k-2} (k-r)(k-r-1) 2^{6(k-r-2)} h_r(T_{\mathbf{C}}M),
\end{aligned}$$

which is just (2.17).

To prove Theorem 2.2 for $8k$ dimensional case, similarly we set (cf. [16])

$$(5.27) \quad P_1(\tau) = \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch} \left(\Theta_1(T_{\mathbf{C}}M), \nabla^{\Theta_1(T_{\mathbf{C}}M)} \right) \right\}^{(8k)},$$

$$(5.28) \quad P_2(\tau) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} \left(\Theta_2(T_{\mathbf{C}}M), \nabla^{\Theta_2(T_{\mathbf{C}}M)} \right) \right\}^{(8k)}.$$

Then one similarly finds that $P_1(\tau)$ is a modular form of weight $4k$ over $\Gamma_0(2)$; while $P_2(\tau)$ is a modular form of weight $4k$ over $\Gamma^0(2)$ and

$$(5.29) \quad P_1(-1/\tau) = (2\tau)^{4k} P_2(\tau).$$

This time, applying Lemma 5.1, we have

$$(5.30) \quad P_2(\tau) = z_0(T_{\mathbf{C}}M) (8\delta_2)^{2k} + z_1(T_{\mathbf{C}}M) (8\delta_2)^{2k-2} \varepsilon_2 + \cdots + z_k(T_{\mathbf{C}}M) \varepsilon_2^k.$$

And thus

$$(5.31) \quad P_1(\tau) = 2^{4k} [z_0(T_{\mathbf{C}}M)(8\delta_1)^{2k} + z_1(T_{\mathbf{C}}M)(8\delta_1)^{2k-2}\varepsilon_1 + \cdots + z_k(T_{\mathbf{C}}M)\varepsilon_1^k].$$

Now we have,

$$(5.32) \quad z_0(T_{\mathbf{C}}M) = \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(8k)},$$

$$(5.33) \quad z_1(T_{\mathbf{C}}M) = - \left\{ \widehat{A}(TM, \nabla^{TM}) \left[48k - \text{ch} \left(B_1(T_{\mathbf{C}}M), \nabla^{B_1(T_{\mathbf{C}}M)} \right) \right] \right\}^{(8k)}.$$

Expanding $\Theta_1(T_{\mathbf{C}}M)$ explicitly, we have

$$(5.34) \quad \begin{aligned} \Theta_1(T_{\mathbf{C}}M) &= \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbf{C}}M) \Lambda_{-q^n}(\mathbf{C}^{8k}) \\ &\quad \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(T_{\mathbf{C}}M) S_{-q^m}(\mathbf{C}^{8k}) \\ &= (1 + (T_{\mathbf{C}}M)q + (S^2 T_{\mathbf{C}}M)q^2 + \cdots)(1 + (T_{\mathbf{C}}M)q^2 + \cdots) \\ &\quad (1 - \mathbf{C}^{8k}q + (\Lambda^2 \mathbf{C}^{8k})q^2 + \cdots)(1 - \mathbf{C}^{8k}q^2 + \cdots) \\ &\quad (1 + (T_{\mathbf{C}}M)q + (\Lambda^2 T_{\mathbf{C}}M)q^2 + \cdots)(1 + (T_{\mathbf{C}}M)q^2 + \cdots) \\ &\quad (1 - \mathbf{C}^{8k}q + (S^2 \mathbf{C}^{8k})q^2 + \cdots)(1 - \mathbf{C}^{8k}q^2 + \cdots) \\ &= (1 + 2(T_{\mathbf{C}}M)q + (T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + S^2 T_{\mathbf{C}}M + \Lambda^2 T_{\mathbf{C}}M)q^2 + \cdots) \\ &\quad (1 + 2(T_{\mathbf{C}}M)q^2 + \cdots) \\ &\quad (1 - 2\mathbf{C}^{8k}q + (\mathbf{C}^{8k} \otimes \mathbf{C}^{8k} + S^2 \mathbf{C}^{8k} + \Lambda^2 \mathbf{C}^{8k})q^2 + \cdots) \\ &\quad (1 - 2\mathbf{C}^{8k}q^2 + \cdots) \\ &= (1 + 2(T_{\mathbf{C}}M)q + 2(T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M)q^2 + \cdots) \\ &\quad (1 - 2\mathbf{C}^{8k}q + 2(\mathbf{C}^{8k} \otimes \mathbf{C}^{8k} - \mathbf{C}^{8k})q^2 + \cdots) \\ &= 1 + 2(T_{\mathbf{C}}M - \mathbf{C}^{8k})q \\ &\quad + 2[-(16k - 1)T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + 8k(8k - 1)]q^2 + \cdots, \end{aligned}$$

where the “...” are the terms involving q^j 's with $j \geq 3$. Note that

$$(5.35) \quad \begin{aligned} &(8\delta_1)^{2k-2r} \varepsilon_1^r \\ &= (2 + 48q + 48q^2 \cdots)^{2k-2r} \left(\frac{1}{16} - q + 7q^2 \cdots \right)^r \\ &= 2^{2k-6r} [1 + 24(2k - 2r)q + 24(k - r)(48k - 48r - 22)q^2 \cdots] \\ &\quad [1 - 16rq + 16(8r^2 - r)q^2 \cdots] \\ &= 2^{2k-6r} [1 + (48k - 64r)q + \\ &\quad (1152k^2 - 3072kr + 2048r^2 - 528k + 512r)q^2 + \cdots]. \end{aligned}$$

Therefore by (5.27), (5.34) and (5.35), comparing the constant terms of both sides of (5.31), we get the result of Liu ([16])

$$(5.36) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k)} = \sum_{r=0}^k 2^{6k-6r} z_r(T_{\mathbf{C}}M).$$

By (5.27), (5.34) and (5.35), comparing the coefficients of q of both sides of (5.31), we have

$$(5.37) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch} (2(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - 2(8k)) \right\}^{(8k)} = \sum_{r=0}^k 2^{6k-6r} (48k - 64r) z_r(T_{\mathbf{C}}M).$$

Thus

$$(5.38) \quad \begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch} ((T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - 8k) \right\}^{(8k)} \\ &= \sum_{r=0}^k 2^{6k-6r} (24k - 32r) z_r(T_{\mathbf{C}}M) \\ &= -8k \left(\sum_{r=0}^k 2^{6k-6r} z_r(T_{\mathbf{C}}M) \right) + \sum_{r=0}^k 2^{6k-6r} (32k - 32r) z_r(T_{\mathbf{C}}M) \\ &= -8k \left(\sum_{r=0}^k 2^{6k-6r} z_r(T_{\mathbf{C}}M) \right) + 32 \cdot 2^6 \sum_{r=0}^{k-1} (k-r) 2^{6k-6r-6} z_r(T_{\mathbf{C}}M). \end{aligned}$$

Combining (5.36) and (5.38), we get

$$(5.39) \quad \begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) \right\}^{(8k)} \\ &= 2^5 \left[\sum_{r=0}^k (k-r) 2^{6(k-r)} z_r(T_{\mathbf{C}}M) \right] \\ &= 2^{11} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} z_r(T_{\mathbf{C}}M) \right], \end{aligned}$$

which is just (2.23).

By (5.27), (5.34) and (5.35), comparing the coefficients of q^2 of both sides of (5.31), we have

$$\begin{aligned} & \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch} (2[-(16k-1)T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + 8k(8k-1)]) \right\}^{(8k)} \\ &= \sum_{r=0}^k z_r(T_{\mathbf{C}}M) 2^{6k-6r} (1152k^2 - 3072kr + 2048r^2 - 528k + 512r). \end{aligned}$$

Thus, combining (5.36) and (5.39), we have

$$\begin{aligned}
& \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}([-(16k-1)T_{\mathbf{C}}M + T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M + 8k(8k-1)]) \right\}^{(8k)} \\
&= \sum_{r=0}^k z_r(T_{\mathbf{C}}M) 2^{6k-6r} (576k^2 - 1536kr + 1024r^2 - 264k + 256r) \\
&= -16k \cdot 2^5 \left[\sum_{r=0}^k (k-r) 2^{6(k-r)} z_r(T_{\mathbf{C}}M) \right] + 24 \cdot 2^5 \left[\sum_{r=0}^k (k-r) 2^{6(k-r)} z_r(T_{\mathbf{C}}M) \right] \\
&\quad + (64k^2 - 8k) \sum_{r=0}^k 2^{6k-6r} z_r(T_{\mathbf{C}}M) \\
&\quad + 2^{10} \sum_{r=0}^k (k-r)(k-r-1) 2^{6(k-r)} z_r(T_{\mathbf{C}}M) \\
&= -16k \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8k)} \\
&\quad + 24 \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8k)} \\
&\quad + (64k^2 - 8k) \left\{ \widehat{L}(TM, \nabla^{T_{\mathbf{C}}M}) \right\}^{(8k+4)} \\
&\quad + 2^{10} \sum_{r=0}^k (k-r)(k-r-1) 2^{6(k-r)} z_r(T_{\mathbf{C}}M).
\end{aligned}$$

Therefore by above computations, we have

$$\begin{aligned}
& \left\{ \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M \otimes T_{\mathbf{C}}M) - 23 \widehat{L}(TM, \nabla^{TM}) \text{ch}(T_{\mathbf{C}}M) \right\}^{(8k)} \\
&= 2^{10} \sum_{r=0}^k (k-r)(k-r-1) 2^{6(k-r)} z_r(T_{\mathbf{C}}M) \\
&= 2^{22} \sum_{r=0}^{k-2} (k-r)(k-r-1) 2^{6(k-r-2)} z_r(T_{\mathbf{C}}M),
\end{aligned}$$

which is just (2.26).

5.2. Proof of Theorem 2.5 and 2.6. The proof for the cases with the extra complex line bundle ξ involved is similar to the above proof.

For $\tau \in \mathbf{H}$ and $q = e^{2\pi\sqrt{-1}\tau}$, set (cf. [8, 9])

$$(5.40) \quad P_1(\xi_{\mathbf{C}}, \tau) = \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \text{ch}\left(\Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}), \nabla^{\Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})}\right) \right\}^{(8k+4)},$$

(5.41)

$$P_2(\xi_{\mathbf{C}}, \tau) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}\left(\Theta_2(T_{\mathbf{C}}M, \xi_{\mathbf{C}}), \nabla^{\Theta_2(T_{\mathbf{C}}M, \xi_{\mathbf{C}})}\right) \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)},$$

where $\nabla^{\Theta_i(T_{\mathbf{C}}M, \xi_{\mathbf{C}})}, i = 1, 2$, are the Hermitian connections with $q^{j/2}$ -coefficients on $\Theta_i(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$ induced from those on the $A_j(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$'s and $B_j(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$'s.

Let $\{\pm 2\pi\sqrt{-1}x_j\}$ be the formal Chern roots for $(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M})$, $c = 2\pi\sqrt{-1}u$. In terms of the theta-functions, we get (cf. [9])

(5.42)

$$P_1(\xi_{\mathbf{C}}, \tau) = 2^{4k+2} \left\{ \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)} \right) \frac{\theta_1^2(0, \tau)}{\theta_1^2(u, \tau)} \frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \frac{\theta_2(u, \tau)}{\theta_2(0, \tau)} \right\}^{(8k+4)},$$

(5.43)

$$P_2(\xi_{\mathbf{C}}, \tau) = \left\{ \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_2(x_j, \tau)}{\theta_2(0, \tau)} \right) \frac{\theta_2^2(0, \tau)}{\theta_2^2(u, \tau)} \frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \frac{\theta_1(u, \tau)}{\theta_1(0, \tau)} \right\}^{(8k+4)}.$$

Applying the transformation laws (5.6) to (5.9) for theta-functions, we still see that ([9]) $P_1(\xi_{\mathbf{C}}, \tau)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$; while $P_2(\xi_{\mathbf{C}}, \tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Moreover, the following identity holds,

(5.44)

$$P_1(\xi_{\mathbf{C}}, -1/\tau) = (2\tau)^{4k+2} P_2(\xi_{\mathbf{C}}, \tau).$$

Then similar to (5.17), we have

(5.45)

$$P_2(\xi_{\mathbf{C}}, \tau) = h_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_2)^{2k+1} + h_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_2)^{2k-1}\varepsilon_2 + \cdots + h_k(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_2)\varepsilon_2^k,$$

where each $h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$, $0 \leq r \leq k$, can be expressed through a canonical integral linear combination of $\left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(B_j(T_{\mathbf{C}}M, \xi_{\mathbf{C}}), \nabla^{B_j(T_{\mathbf{C}}M, \xi_{\mathbf{C}})}) \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)}$, $0 \leq j \leq r$.

Explicitly, one has ([9])

(5.46)

$$h_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = - \left\{ \widehat{A}(TM, \nabla^{TM}) \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)},$$

(5.47)

$$h_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \left\{ \widehat{A}(TM, \nabla^{TM}) [24(2k+1) - \text{ch}(B_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}))] \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)}.$$

By (5.44) and (5.45), we have

(5.48)

$$P_1(\xi_{\mathbf{C}}, \tau) = 2^{4k+2} [h_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_1)^{2k+1} + h_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_1)^{2k-1}\varepsilon_1 + \cdots + h_k(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_1)\varepsilon_1^k].$$

Let's explicitly expand $\Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$. By (2.5) and (2.8), we have

$$\begin{aligned}
 (5.49) \quad \Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbf{C}}M} - 2\widetilde{\xi_{\mathbf{C}}}) \\
 &\quad \otimes \bigotimes_{r=1}^{\infty} \left(\Lambda_{q^{r-\frac{1}{2}}}(\xi_{\mathbf{C}}) S_{-q^{r-\frac{1}{2}}}(\mathbf{C}^2) \right) \otimes \left(\bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\xi_{\mathbf{C}}) S_{q^{s-\frac{1}{2}}}(\mathbf{C}^2) \right) \\
 &= (1 + (T_{\mathbf{C}}M - (8k+4))q) \otimes (1 + (T_{\mathbf{C}}M - (8k+4) - 2\xi_{\mathbf{C}} + 4)q) \\
 &\quad \otimes (1 + (\xi_{\mathbf{C}})q^{\frac{1}{2}} + (\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}})q) \otimes (1 - \mathbf{C}^2 q^{\frac{1}{2}} + \mathbf{C}^3 q) \\
 &\quad \otimes (1 - (\xi_{\mathbf{C}})q^{\frac{1}{2}} + (\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}})q) \otimes (1 + \mathbf{C}^2 q^{\frac{1}{2}} + \mathbf{C}^3 q) + \dots \\
 &= 1 + [2(T_{\mathbf{C}}M - (8k+4) - \xi_{\mathbf{C}} + 2) - (\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2)]q + \dots,
 \end{aligned}$$

where the “...” terms are the terms involving $q^{\frac{j}{2}}$'s with $j \geq 3$.

By (5.22), (5.40) and (5.49), setting $q = 0$ in (5.48), we have ([9])

$$(5.50) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8k+4)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

On the other hand, by (5.22), (5.40) and (5.49), comparing the coefficients of q in (5.48), we have

$$\begin{aligned}
 (5.51) \quad &\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \text{ch} \left(2(T_{\mathbf{C}}M) - 2(8k+4) - 2(\xi_{\mathbf{C}} - 2) - (\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2) \right) \right\}^{(8k+4)} \\
 &= 8 \sum_{r=0}^k 2^{6k-6r} (48k + 24 - 64r) h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).
 \end{aligned}$$

Thus similar to (5.25), we have

$$\begin{aligned}
 (5.52) \quad &\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \text{ch} \left(T_{\mathbf{C}}M - (8k+4) - (\xi_{\mathbf{C}} - 2) - \frac{1}{2}(\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2) \right) \right\}^{(8k+4)} \\
 &= 12 \left(8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right) - 8k \left(8 \sum_{r=0}^k 2^{6k-6r} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right) \\
 &\quad + 256 \cdot 2^6 \sum_{r=0}^{k-1} (k-r) h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) 2^{6k-6r-6}.
 \end{aligned}$$

Note that

$$\begin{aligned}
(5.53) \quad & \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \text{ch} \left((\xi_{\mathbf{C}} - 2) + \frac{1}{2}(\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2) \right) \\
&= \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \left((e^c + e^{-c} - 2) + \frac{1}{2}((e^c + e^{-c})^2 - 4) \right) \\
&= \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \sinh^2\left(\frac{c}{2}\right) (2(e^c + e^{-c}) + 8) \\
&= \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \sinh^2\left(\frac{c}{2}\right) \text{ch}(2\xi_{\mathbf{C}} \oplus \mathbf{C}^8).
\end{aligned}$$

Therefore combining (5.50), (5.52) and (5.53), one has

$$\begin{aligned}
(5.54) \quad & \left\{ \frac{\widehat{L}(TM, \nabla^{TM}) [\text{ch}(T_{\mathbf{C}}M) - \sinh^2\left(\frac{c}{2}\right) \text{ch}(2\xi_{\mathbf{C}} \oplus \mathbf{C}^8) - 16]}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8k+4)} \\
&= 2^{14} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} h_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right],
\end{aligned}$$

which is just (2.31).

To prove Theorem 2.6 for $8k$ dimensional case, similarly we set ([9])

$$(5.55) \quad P_1(\xi_{\mathbf{C}}, \tau) = \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)} \text{ch} \left(\Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}), \nabla^{\Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})} \right) \right\}^{(8k)},$$

(5.56)

$$P_2(\xi_{\mathbf{C}}, \tau) = \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} \left(\Theta_2(T_{\mathbf{C}}M, \xi_{\mathbf{C}}), \nabla^{\Theta_2(T_{\mathbf{C}}M, \xi_{\mathbf{C}})} \right) \cosh\left(\frac{c}{2}\right) \right\}^{(8k)}.$$

Still playing the same game, we see that $P_1(\xi_{\mathbf{C}}, \tau)$ is a modular form of weight $4k$ over $\Gamma_0(2)$; while $P_2(\xi_{\mathbf{C}}, \tau)$ is a modular form of weight $4k$ over $\Gamma^0(2)$ and one has the following identities,

$$(5.57) \quad P_1(-1/\tau) = (2\tau)^{4k} P_2(\tau),$$

(5.58)

$$P_2(\tau) = z_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_2)^{2k} + z_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_2)^{2k-2}\varepsilon_2 + \cdots + z_k(T_{\mathbf{C}}M, \xi_{\mathbf{C}})\varepsilon_2^k.$$

Thus

(5.59)

$$P_1(\tau) = 2^{4k} [z_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_1)^{2k} + z_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})(8\delta_1)^{2k-2}\varepsilon_1 + \cdots + z_k(T_{\mathbf{C}}M, \xi_{\mathbf{C}})\varepsilon_1^k].$$

By direct computations, we have

$$(5.60) \quad z_0(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = \left\{ \widehat{A}(TM, \nabla^{TM}) \right\}^{(8k)},$$

$$(5.61) \quad z_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = - \left\{ \widehat{A}(TM, \nabla^{TM}) [48k - \text{ch}(B_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}))] \right\}^{(8k)}.$$

As we did in (5.49), explicitly expanding $\Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}})$, we get

$$(5.62) \quad \Theta_1(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) = 1 + [2(T_{\mathbf{C}}M - (8k) - \xi_{\mathbf{C}} + 2) - (\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2)]q + \cdots,$$

where the “...” terms are the terms involving $q^{\frac{j}{2}}$'s with $j \geq 3$.

By (5.35), (5.55) and (5.62), setting $q = 0$ in (5.59), we have ([8, 9])

$$(5.63) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \right\}^{(8k)} = \sum_{r=0}^k 2^{6k-6r} z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

By (5.35), (5.55) and (5.62), comparing the coefficients of q in (5.59), we have

$$(5.64) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch}(2(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - 2 \cdot 8k - 2(\xi_{\mathbf{C}} - 2) - (\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2)) \right\}^{(8k)} \\ = \sum_{r=0}^k 2^{6k-6r} (48k - 64r) z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

Thus similar to (5.38), we have

$$(5.65) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} \text{ch} \left((T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - 8k - (\xi_{\mathbf{C}} - 2) - \frac{1}{2}(\xi_{\mathbf{C}} \otimes \xi_{\mathbf{C}} - 2\xi_{\mathbf{C}} \wedge \xi_{\mathbf{C}} - \mathbf{C}^2) \right) \right\}^{(8k)} \\ = -8k \left(\sum_{r=0}^k 2^{6k-6r} z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right) + 32 \cdot 2^6 \sum_{r=0}^{k-1} (k-r) 2^{6k-6r-6} z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}).$$

Combining (5.63), (5.65) and (5.53), one has

$$(5.66) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM}) [\text{ch}(T_{\mathbf{C}}M, \nabla^{T_{\mathbf{C}}M}) - \sinh^2(\frac{c}{2}) \text{ch}(2\xi_{\mathbf{C}} \oplus \mathbf{C}^8)]}{\cosh^2(\frac{c}{2})} \right\}^{(8k)} \\ = 2^{11} \left[\sum_{r=0}^{k-1} (k-r) 2^{6(k-r-1)} z_r(T_{\mathbf{C}}M, \xi_{\mathbf{C}}) \right],$$

Which is just (2.32).

Remark 5.1. *Our main results are obtained by comparing coefficients of q and q^2 in (5.20), (5.31) and coefficients of q in (5.48), (5.59). It is interesting to examine other coefficients of higher power of q to get further divisibility and congruence results. These will be developed elsewhere.*

6. ACKNOWLEDGMENTS

We are grateful to Professor Weiping Zhang for helpful suggestions and inspiring discussions with us. Professor M. Atiyah and Pofessor F. Hirzebruch are deeply appreciated for communications with us. We also thank Professor Peter Teichner and Professor Nicolai Reshetikhin for their interests, encouragements and many helpful discussions.

REFERENCES

1. L. Alvarez-Gaumé and E. Witten, Gravitational anomalies. *Nucl. Phys.* B234 (1983), 269-330.
2. M. F. Atiyah, *K - theory*. Benjamin, New York, 1967.
3. M. F. Atiyah and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.* 65 (1959), 276-281.

4. M.F. Atiyah and I.M. Singer, The index of elliptic operators, III, *Ann. Math.* 87 (1968), 546-604.
5. K. Chandrasekharan, *Elliptic Functions*. Springer-Verlag, 1985.
6. S. M. Finashin, A Pin^- -cobordism invariant and a generalization of Rokhlin signature congruence. *Leningrad Math. J.* 2 (1991), 917-924.
7. Peter B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, Second Edition. CRC Press, Inc, 1995.
8. F. Han and W. Zhang, Spin^c -manifold and elliptic genera. *C. R. Acad. Sci. Paris, Série I.* 336 (2003), 1011-1014.
9. F. Han and W. Zhang, Modular invariance, characteristic numbers and η invariants. *Journal of Differential Geometry.* 67 (2004), 257-288.
10. F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*. Aspects of Mathematics, vol. E20, Vieweg, Braunschweig, 1992.
11. F. Hirzebruch, *Topological Methods in Algebraic Geometry*. Springer-Verlag, 1966.
12. F. Hirzebruch, Mannigfaltigkeiten und Modulformen. *Jahresberichte der Deutschen Mathematiker Vereinigung*, Jber. d. Dt. Math.-Verein, 1992, pp. 20-38.
13. Boyuan Hou and Boyu Hou, *Differential Geometry for Physicists*, Second Edition (in Chinese). Science Press, China, 2004.
14. P. S. Landweber, Elliptic cohomology and modular forms. in *Elliptic Curves and Modular Forms in Algebraic Topology*, p. 55-68. Ed. P. S. Landweber. Lecture Notes in Mathematics Vol. 1326, Springer-Verlag (1988).
15. G. Laures, $K(1)$ -local topological modular forms. *Invent Math.* (2004), 371-403.
16. K. Liu, Modular invariance and characteristic numbers. *Commun. Math. Phys.* 174 (1995), 29-42.
17. K. Liu, On Modular Invariance and Rigidity Theorems, *Ph.D Dissertation at Harvard University*. 1993
18. K. Liu and W. Zhang, Elliptic genus and η -invariants. *Inter. Math. Res. Notices* No. 8 (1994), 319-328.
19. S. Ochanine, Signature modulo 16, invariants de Kervaire généralisés et nombre caractéristiques dans la K -théorie réelle. *Mémoire Soc. Math. France*, Tom. 109 (1987), 1-141.
20. W. Zhang, Spin^c -manifolds and Rokhlin congruences. *C. R. Acad. Sci. Paris, Série I*, 317 (1993), 689-692.
21. W. Zhang, Circle bundles, adiabatic limits of η invariants and Rokhlin congruences. *Ann. Inst. Fourier* 44 (1994), 249-270.
22. W. Zhang, *Lectures on Chern-Weil Theory and Witten Deformations*. Nankai Tracts in Mathematics Vol. 4, World Scientific, Singapore, 2001.

Q. CHEN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA, 94720-3840

E-mail address: chenqtao@math.berkeley.edu

F. HAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA, 94720-3840

E-mail address: feihan@math.berkeley.edu